SOME NEW NONIMMERSION RESULTS FOR REAL PROJECTIVE SPACES

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ABSTRACT. We use the spectrum tmf to obtain new nonimmersion results for many real projective spaces RP^n for n as small as 113. The only new ingredient is some new calculations of tmf-cohomology groups. We present an expanded table of nonimmersion results. Our new theorem is new for 17% of the values of n between 2^i and $2^i + 2^{14}$ for $i \geq 15$.

1. INTRODUCTION

We use the spectrum tmf to prove the following new nonimmersion theorem for real projective spaces P^n.

Theorem 1.1. *Let $\alpha(n)$ denote the number of 1's in the binary expansion of n.*

a. *If $\alpha(M) = 3$, then P^{8M+9} does not immerse in ($\not\subseteq$) $\mathbb{R}^{16M-1}$.*
b. *If $\alpha(M) = 6$, then $P^{8M+9} \not\subseteq \mathbb{R}^{16M-11}$.*
c. *If $\alpha(M) = 7$, then $P^{16M+16} \not\subseteq \mathbb{R}^{32M-7}$ and $P^{16M+17} \not\subseteq \mathbb{R}^{32M-6}$.*
d. *If $\alpha(M) = 9$, then $P^{32M+25} \not\subseteq \mathbb{R}^{64M-4}$ and $P^{32M+26} \not\subseteq \mathbb{R}^{64M-3}$.*
e. *If $\alpha(M) = 10$, then $P^{16M+17} \not\subseteq \mathbb{R}^{32M-20}$ and $P^{16M+18} \not\subseteq \mathbb{R}^{32M-19}$.*

We apply the same method that was used in [4], using $\mathrm{tmf}^*(-)$ to detect nonexistence of axial maps. The novelty here is that we compute and utilize groups $\mathrm{tmf}^*(P^m \wedge P^n)$ when m and/or n is odd. In [4], only even values of m and n were studied. There is, however, no significant difference or complication in using the odd values. We prove Theorem 1.1 in Section 2.

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For many years, the author has maintained a website ([5]) which listed all known immersion, nonimmersion, embedding, and nonembedding results for P^n and tabulated them for $n = 2^i + d$ with $2^i > d$ and $0 \le d \le 63$. In [12], W. Stephen Wilson acknowledged how this table motivated him to try (and succeed) to prove nonimmersions for small P^n. Our Theorem 1.1(a.) includes $P^{2^i+49} \not\subseteq \mathbb{R}^{2^{i+1}+79}$ and $P^{2^i+57} \not\subseteq \mathbb{R}^{2^{i+1}+95}$ for $i \ge 6$, which improve on previous best results (of [12]) by 1 and 2 dimensions, respectively, and hence enter the table [5].

To facilitate checking whether results are new, the author has greatly expanded his table of nonimmersion results at `www.lehigh.edu/~dmd1/imms.html`. We have listed there the best known nonimmersions for P^{2^i+d} for $2^i > d$ and $0 \le d \le 16{,}383$ together with the first acknowledged source. A listing of and link to the `Maple` program that generated this table is also included there. This table gives all known nonimmersion results for P^n with $7 < n < 49{,}152$ except for James' nonimmersions of P^{2^e-1} in dimension $2^{e+1} - 2e - \langle 3, 2, 2, 4\rangle$ if $e \equiv \langle 0, 1, 2, 3\rangle \bmod 4$.([11])

Theorem 1.1 appears 2796 times in this table, thus giving new results for 17% of the projective spaces of dimension between 2^i and $2^i + 2^{14}$ for $i \ge 15$. The seminal result of [6],

$$P^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{4m-2\alpha(m)}, \tag{1.2}$$

appears 7063 times in the table, but is divided among four references. The first 4361 of them appeared in [1], which obtained a result equivalent to (1.2) for P^n with n satisfying a very complicated condition. The statement (1.2) was first conjectured in [2] and proved there for $\alpha(m) \le 6$, which yielded 168 new results in this table. It was extended to $\alpha(m) = 7$ in [13], and this still applies to 700 values. This left 1834 values which were covered by the general result (1.2) and not by any of the three preceding references, and have not been bettered in subsequent work.

The first tmf-paper, ([4]), appears 2866 times in the table; there are 110 additional values for small $\alpha(-)$ of tmf-implied nonimmersions which were overlooked in [4] and noted in [7]. The other big collection of nonimmersion results is those obtained in [12] using $ER(2)$-cohomology, which appears 2092 times. Both $ER(2)$ and tmf can be considered as real-versions of $BP\langle 2\rangle$. Using $ER(2)$ is advantageous because $ER(2)^*(P^n)$ has a 2-dimensional class, while $\mathrm{tmf}^*(P^n)$ only has an 8-dimensional class. Also $ER(2)$ is more closely related to $BP\langle 2\rangle$, and so, as W. Stephen Wilson

says, it can "mooch" off the result (1.2). The advantage of tmf is that some of its groups are one 2-power larger than those of $ER(2)$.

In [6], it was stated that (1.2) was within 2 dimensions of all known nonimmersion results, in the sense that the two dimensions could come from the Euclidean space, the projective space, or a combination. In other words, if $D(n)$ denotes the nonimmersion dimension for P^n obtained from (1.2), and $K(n)$ the best known nonimmersion dimension for P^n, then, at the time, it was true that

$$K(n) \leq \max(D(n)+2,\ D(n+1)+1,\ D(n+2)). \tag{1.3}$$

This is no longer true. There are 10 values of n in the table for which the result of [9], which states that if $\alpha(n)=4$ and $n \equiv 10 \bmod 16$ then $P^n \not\subseteq \mathbb{R}^{2n-9}$, does not satisfy (1.3), and there are 418 values of n in the table for which Theorem 1.1(c) does not satisfy (1.3). These are the only results which are more than 2 stronger than (1.2) in the sense of (1.3), and it is still true that (1.2) is within 3 dimensions of all known results in the same sense. That is, the following statement is currently true.

$$K(n) \leq \max(D(n)+3,\ D(n+1)+2,\ D(n+2)+1,\ D(n+3)).$$

The first example of (1.3) not being satisfied occurs for $n=58$; we have $K(58)=107$ due to [9] (which used modified Postnikov towers) while $D(58)=D(59)=98$ and $D(60)=D(61)=106$. The first example of our 1.1(c) causing (1.3) to be not satisfied occurs from $K(3584)=7129$ (due to 1.1(c)) while $D(3584)=D(3585)=7124$ and $D(3586)=D(3587)=7128$.

Theorem 1.1 can be extended to larger values of $\alpha(M)$ similarly to what was done in [4]. We have emphasized the results for small values of $\alpha(M)$ for clarity of exposition. The extension, whose proof we sketch in Section 3, is as follows. The lettering of the parts corresponds to the parts of Theorem 1.1.

Theorem 1.4. *Let $p(h)$ denote the smallest 2-power $\geq h$.*

- b,e. *Suppose $\alpha(M)=4h+2$ and $h \leq 2^{e_1}-2^{e_0}$ if $M \equiv 2^{e_0}+2^{e_1} \bmod 2^{e_1+1}$ with $e_0 < e_1$. Then*
  - b. *If h is odd, $P^{8M+8h+1} \not\subseteq \mathbb{R}^{16M-8h-3}$, and*
  - e. *If h is even, then $P^{8M+8h+1} \not\subseteq \mathbb{R}^{16M-8h-4}$ and $P^{8M+8h+2} \not\subseteq \mathbb{R}^{16M-8h-3}$.*

c. *If* $\alpha(M) = 4h+3$ *with* h *odd and* $M \equiv 0 \mod p(h+1)$*, then* $P^{8M+8h+8} \not\subseteq \mathbb{R}^{16M-8h+1}$ *and* $P^{8M+8h+9} \not\subseteq \mathbb{R}^{16M-8h+2}$*.*

d. *If* $\alpha(M) = 4h+1$ *with* h *even and* $M \equiv 0 \mod p(h+1)$*, then* $P^{8M+8h+9} \not\subseteq \mathbb{R}^{16M-8h+12}$ *and* $P^{8M+8h+10} \not\subseteq \mathbb{R}^{16M-8h+13}$*.*

2. Proof of Theorem 1.1

Let tmf denote the 2-local connective spectrum introduced in [10], whose mod-2 cohomology is the quotient of the mod-2 Steenrod algebra A by the left ideal generated by Sq^1, Sq^2, and Sq^4. Thus $\mathrm{tmf}_*(X)$ may be computed by the Adams spectral sequence (ASS) with $E_2 = \mathrm{Ext}_{A_2}(H^*X, \mathbb{Z}_2)$, where A_2 is the subalgebra of A generated by Sq^1, Sq^2, and Sq^4. We rely on Bob Bruner's software ([3]) for our calculations of these Ext groups. It was proved in [7, p.167] that there are 8-dimensional classes X, X_1, and X_2 such that the homomorphism in $\mathrm{tmf}^*(-)$ induced by an axial map $P^m \times P^n \to P^k$ effectively sends X to $u(X_1 + X_2)$, where u is a unit in $\mathrm{tmf}^0(P^m \times P^n)$ which will be omitted from our exposition.

We will often use duality isomorphisms $\mathrm{tmf}^i(P^n) \approx \mathrm{tmf}_{-i-1}(P_{-n-1})$ for $i > 2$, and $\mathrm{tmf}^i(P^m \wedge P^n) \approx \mathrm{tmf}_{-i-2}(P_{-m-1} \wedge P_{-n-1})$ for $i > \max(m,n) + 2$. For any integer m, P_m denotes the spectrum P_m^∞. We make frequent use of the periodicity $P_{b+8}^{t+8} \wedge \mathrm{tmf} \simeq \Sigma^8 P_b^t \wedge \mathrm{tmf}$ proved in [4, Prop 2.6].

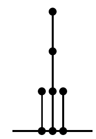
We let $\nu(-)$ denote the exponent of 2 in an integer, and use $\nu\left(\binom{m}{n}\right) = \alpha(n) + \alpha(m-n) - \alpha(n)$. Also, if L is large, $\nu\left(\binom{2^L-k}{n}\right) = \nu\left(\binom{-k}{n}\right) = \nu\left(\binom{n+k-1}{n}\right)$. We will never be interested in the values of odd factors of coefficients, and will not list them.

Proof of (a). If the immersion exists, there is an axial map $P^{8M+9} \times P^{8M+9} \to P^{16M-1}$. The induced homomorphism in $\mathrm{tmf}^*(-)$ sends $0 = X^{2M}$ to

$$\sum \tbinom{2M}{i} X_1^i X_2^{2M-i} \tag{2.1}$$

in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. This group is isomorphic to $\mathrm{tmf}_{-2}(P_{-10} \wedge P_{-10}) \approx \mathrm{tmf}_{30}(P_6 \wedge P_6)$. The portion of the ASS for $\mathrm{tmf}_{30}(P_6 \wedge P_6)$ arising from filtration 0 by h_0-extensions appears in Diagram 2.2.

Diagram 2.2. *Portion of* $\mathrm{tmf}_{30}(P_6 \wedge P_6)$



There are several elements in higher filtration which are not relevant to our argument. The elements pictured in Diagram 2.2 cannot be hit by differentials in the ASS because in dimension 31 there is only one tower in low enough filtration and it cannot support a differential by the argument of [4, p.54], namely that its generator is a constructible homotopy class. The filtration-0 elements must correspond to $X_1^{M-1}X_2^{M+1}$, $X_1^M X_2^M$, and $X_1^{M+1}X_2^{M-1}$ in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. Since

$$2^2 X^{M+1} = 0 \text{ in } \mathrm{tmf}^*(P^{8M+9}), \tag{2.3}$$

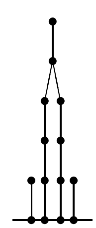
the two $\mathbb{Z}/4$'s in Diagram 2.2 must represent $X_1^{M\pm1}X_2^{M\mp1}$, and multiples of these are 0 in all filtrations > 1. Thus $X_1^M X_2^M$ generates the $\mathbb{Z}/2^4$ in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. Since $\nu\left(\binom{2M}{M}\right) = \alpha(M) = 3$, we obtain that (2.1) is nonzero, contradicting the existence of the immersion. □

Proof of (b). If the immersion exists, there is an axial map $P^{8M+9} \times P^{2^{L+3}-16M+9} \to P^{2^{L+3}-8M-11}$ for sufficiently large L. Hence

(2.4)
$$\sum \binom{-M-1}{i} X_1^i X_2^{2^L-M-1-i} = 0 \in \mathrm{tmf}^{2^{L+3}-8M-8}(P^{8M+9} \wedge P^{2^{L+3}-16M+9}).$$

This group is isomorphic to $\mathrm{tmf}_{38}(P_6 \wedge P_6)$, and the relevant part of it is given in Diagram 2.5. Similarly to case (a), and continuing in all remaining cases, it cannot be hit by a differential in the ASS.

Diagram 2.5. *Portion of* $\mathrm{tmf}_{38}(P_6 \wedge P_6)$



The outer ($\mathbb{Z}/4$) generators must correspond to $X_1^{M-2}X_2^{2^L-2M+1}$ and $X_1^{M+1}X_2^{2^L-2M-2}$. (Note that 4 times each of these classes is 0 by (2.3), and so they cannot produce a higher-filtration component impacting the middle summands. This will be the case also for the outer summands in subsequent diagrams.) The inner generators must be $X_1^{M-1}X_2^{2^L-2M}$ and $X_1^MX_2^{2^L-2M-1}$. By [4, Thm 2.7], the class $2^4(X_1^{M-1}X_2^{2^L-2M}+X_1^MX_2^{2^L-2M-1})$ is 0 in filtration 4, although it might be nonzero in filtration 5. This is depicted by the behavior of the chart between filtration 3 and 4. Since $\alpha(M)=6$, the component of these terms in (2.4) is

$$\tbinom{-M-1}{M-1}X_1^{M-1}X_2^{2^L-2M}+\tbinom{-M-1}{M}X_1^MX_2^{2^L-2M-1}=2^5X_1^{M-1}X_2^{2^L-2M}+2^6X_1^MX_2^{2^L-2M-1},$$

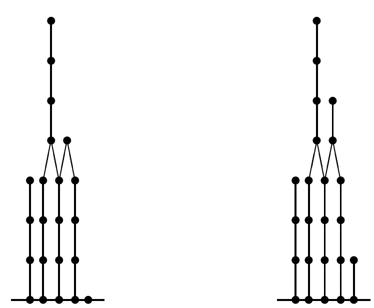
which is nonzero in the group depicted by Diagram 2.5, contradicting the existence of the immersion. □

Proof of (c). If the first immersion exists, there is an axial map $P^{16M+16}\times P^{2^{L+3}-32M+5}\to P^{2^{L+3}-16M-18}$. Hence

(2.6)
$$\sum\tbinom{-2M-2}{i}X_1^iX_2^{2^L-2M-2-i}=0\in \mathrm{tmf}^*(P^{16M+16}\wedge P^{2^{L+3}-32M+5}).$$

This group is isomorphic to $\mathrm{tmf}_{46}(P_7\wedge P_2)$, and the relevant part of it is given in the left side of Diagram 2.7.

Diagram 2.7. *Portion of* $\mathrm{tmf}_{46}(P_7\wedge P_2)$ *and* $\mathrm{tmf}_{46}(P_6\wedge P_3)$



The generators, from left to right, correspond to $X_1^{2M-2}X_2^{2^L-4M},\dots,X_1^{2M+2}X_2^{2^L-4M-4}$, with the sum relation in filtration 4 similar to that of the previous (and future) parts. Since $\alpha(M)=7$, the component of the middle terms in (2.6) is

$$2^{8+\nu(M)}X_1^{2M-1}X_2^{2^L-4M-1}+2^7X_1^{2M}X_2^{2^L-4M-2}+2^8X_1^{2M+1}X_2^{2^L-4M-3},$$

which is nonzero in the group depicted by Diagram 2.7. The argument for the second nonimmersion involves the same sum in a group isomorphic to $\mathrm{tmf}_{46}(P_6 \wedge P_3)$, which is pictured on the right side of Diagram 2.7. $\square$

Proof of (d). The proof is similar to those of parts (b) and (c). The first nonimmersion is proved by showing if $\alpha(M) = 9$, then

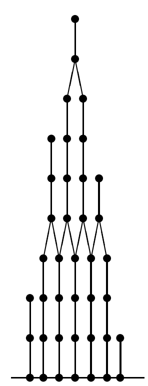
(2.8)
$$\sum \binom{-4M-3}{i} X_1^i X_2^{2^L-4M-3-i} \neq 0 \in \mathrm{tmf}^{2^{L+3}-32M-24}(P^{32M+25} \wedge P^{2^{L+3}-64M+2}).$$

This group is isomorphic to $\mathrm{tmf}_{62}(P_6 \wedge P_5)$, the relevant part of which is depicted in Diagram 2.9, with generators corresponding to $i = 4M-3, \ldots, 4M+3$ in (2.8). The sum relation in filtration 8 follows from [4, Thm 2.7]. The middle components of our class are

$$2^{10+\nu(M)} X_1^{4M-1} X_2^{2^L-8M-2} + 2^9 X_1^{4M} X_2^{2^L-8M-3} + 2^9 X_1^{4M+1} X_2^{2^L-8M-4},$$

which is nonzero in filtration 9. Note that $2^9 X_1^{4M} X_2^{2^L-8M-3}$ is 0 in filtration 9, as can be seen from Diagram 2.9 or from [4, 2.7], which says that if g_1, g_2, g_3 denote the middle three generators, then there are relations that both $2^8(g_1 + g_2 + g_3)$ and $2^8(g_1 + g_3)$ have filtration > 8.

Diagram 2.9. *Portion of* $\mathrm{tmf}_{62}(P_6 \wedge P_5)$



The argument for the second nonimmersion is virtually identical. Its obstruction is the same sum in a group isomorphic to $\mathrm{tmf}_{62}(P_5 \wedge P_6)$, so just the reverse of Diagram 2.9. $\square$

Proof of (e). The obstruction this time is $\sum\binom{-2M-2}{i}X_1^iX_2^{2^L-2M-2-i}$ in a group isomorphic to the one depicted in Diagram 2.9. The middle terms are

$$2^9X_1^{2M-2}X_2^{2^L-4M}+2^{11+\nu(M)}X_1^{2M-1}X_2^{2^L-4M-1}+2^{10}X_1^{2M}X_2^{2^L-4M-2},$$

which is nonzero. □

3. Sketch of proof of Theorem 1.4

We use the v_2^8-periodicity of Ext_{A_2} proved in [8, p.299,Thm 5.9] to see that, if one of the diagrams of Section 2 depicts a portion of $\mathrm{tmf}_i(P_a\wedge P_b)$, then the top part of the portion of $\mathrm{tmf}_{i+48j}(P_a\wedge P_b)$ generated by filtration-0 classes has the same form $8j$ units higher. We also use the arguments on [4, p.54] to see that, when this portion is interpreted as a quotient of a $\mathrm{tmf}^k(P^c\wedge P^d)$ group, the relations are of the same sort as those in [4, Thm 2.7]. The relation [4, (2.10)] is especially important and will be noted specifically below. We use cofiber sequences such as $S^a\wedge P_b\to P_a\wedge P_b\to P_{a+1}\wedge P_b$ to deduce results for our spaces, in which at least one of the bottom dimensions is even, from those of [4], which dealt with the situation when both bottom dimensions are odd. The nice form of $\mathrm{Ext}_{A_2}(H^*P_b)$ below a certain line of slope 1/6 is important here. As noted on [4, p.54], it is just a sum of copies of $\mathrm{Ext}_{A_1}(\mathbb{Z}_2)$, suitably placed.

Proof of 1.4(b,e). If the immersion in (b) exists, there is an axial map

$$P^{8M+8h+1}\times P^{2^{L+3}-16M+8h+1}\to P^{2^{L+3}-8M-8h-3}.$$

We obtain a contradiction to this by showing

(3.1)
$$\sum\binom{-M-h}{i}X_1^iX_2^{2^L-M-h-i}\neq 0\in\mathrm{tmf}^*(P^{8M+8h+1}\wedge P^{2^{L+3}-16M+8h+1}).$$

Our obstruction will be in filtration $4h+1$, where there is a nonzero class by v_2^8-periodicity from Diagram 2.5, which is the case $h=1$. Note that the group in which (3.1) lies is isomorphic to $\mathrm{tmf}_{24h+14}(P_6\wedge P_6)$. The terms in (3.1) with $i>M$ cannot interfere in this filtration because for such i, $2^{4h-2}X_1^i=0$ in $\mathrm{tmf}^*(P^{8M+8h+1})$. The same holds for terms with $i<M-h$ due to the second factor. By [4, 3.12], the coefficients of the terms in (3.1) with $M-h\le i\le M$ are all divisible by $2^{\alpha(M)-1}=2^{4h+1}$. This is where the strange hypothesis comes into play. Next we note

that

$$\nu\Big(\sum_{j=0}^{h}\tbinom{h}{j}\tbinom{-M-h}{M-j}\Big)=\nu\big(\tbinom{-M}{M}\big)=\alpha(M)-1.$$

By a variant on [4, Cor 2.13.3], this implies that (3.1) is nonzero. There are four things that are required to make this work. (a) No interference from the outer terms because they are precisely 0 in a lower filtration. (b) All the $h+1$ intermediate terms have filtration at least $4h+1$. (c) The chart is nonzero in filtration $4h+1$. (d) An odd number of the intermediate terms which have $\binom{h}{j}$ odd, $0\le j\le h$, are nonzero in filtration $4h+1$. This latter is a version of [4, (2.10)]. It is a consequence of a relation in every fourth filtration that the sum of the basic classes in the previous filtration is 0 in that filtration. By "basic," we mean those obtained from canonical classes in filtration 0 or 4 by v_2^8 periodicity.

The proof of (e) is virtually identical. $\square$

Proof of 1.4(c,d). The proof of (d) is virtually identical to that of (c), and this is similar to that of (b) with the main difference being that the obstruction is due to $\binom{-M-1}{M}$ instead of $\binom{-M}{M}$, which causes a very different-looking hypothesis. The contradiction to the first result of (c) is obtained by showing

(3.2)

$$\sum\tbinom{-M-h-1}{i}X_1^iX_2^{2^L-M-h-1-i}\ne 0\in \mathrm{tmf}^*(P^{8M+8h+8}\wedge P^{2^{L+3}-16M+8h-3}).$$

The obstruction will be in filtration $\alpha(M)=4h+3$. The terms with $i>M$ or $i<M-h$ are precisely 0 in filtration less than $4h+3$ due to their first or second factor. By our hypothesis and [4, 3.8], the intermediate terms are all divisible by $2^{\alpha(M)}$. Since

$$\nu\Big(\sum_{j=0}^{h}\tbinom{h}{j}\tbinom{-M-h-1}{M-j}\Big)=\nu\big(\tbinom{-M-1}{M}\big)=\alpha(M),$$

and, by v_2^8-periodicity from Diagram 2.7, the obstruction group is nonzero in filtration $\alpha(M)=4h+3$. $\square$

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